THE INFLUENCE OF OSCILLATIONS OF THE FREE SURFACE OF A LIQUID ON THE STABILITY OF ROTATIONAL MOTION OF A SPINNING TOP CONTAINING THE LIQUID

(VLIIANIE KOLEBANII SVOBODNOI POVERKHNOSTI ZHIDKOSTI NA USTOICHIVOST' VRASHCHATEL'NYKH DVIZHENII VOLCHKA, SODERZHASHCHEGO ZHIDKOSTI)

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B.A. KOSTANDIAN
(Erivan)

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Problems concerning the motion of a rigid body, with a cavity filled with incompressible liquid, have been studied in a very general formulation by Zhukovskii [1]. Chetaev [2], making use of Zhukovskii's work, solved the particular problem of the stability of rotational motion of a projectile with a cylindrical cavity completely filled with ideal incompressible liquid. He also investigated the case when the cylindrical cavity has a diametral diaphragm or a cross (two orthogonal diametral planes). Rumiantsev [3,4] considered the stability of spinning motion of a rigid body having a cavity completely or partially filled with incompressible liquid. He posed the problem of the stability of spinning motion of a rigid body with liquid in relation to every variable characterizing the motion of the rigid body and to some of the variables characterizing the motion of the liquid. Starting from the complete equations of perturbed motion of the whole system, by the method of Liapunov and Chetaev, he obtained sufficient conditions for the stability of the rotational motion of the rigid body.

Sobolev [5] studied the general theory of motion of a symmetrical top with a cavity completely filled with liquid.

The papers of Moiseev [6,7,8] were devoted to the analysis of the case of motion of a rigid body with a partially filled cavity. Assuming the liquid to be ideal and incompressible, the motion of the liquid in a stationary vessel is potential, whilst for the case of small motions of the vessel about the position of equilibrium he obtained a system of differential equations for the motion of the system of vessel and liquid.

Narimanov [9] derived the equations for small motions of a rigid body having a cavity partially filled with liquid, and analysed the solutions of these equations. In another paper Narimanov [10] derived the equations for small perturbations of the steady rotation of a symmetrical gyroscope having a cylindrical cavity partially filled with liquid. The problem led to an infinite system of ordinary differential equations with constant coefficients.

In [11] Stewartson investigated the stability of spinning motion of a top with a partially filled cylindrical cavity. This problem in a somewhat different formulation was studied by the author of the present paper [12]. Both papers consider the linearized equations of the system, and in the computation of the overturning moment acting on the wall at the side of the liquid it is assumed that the free surface of the liquid differs little from the unperturbed cylindrical form. The conditions of stability obtained by Stewartson [11] were checked experimentally. The description of the experiment and the recording of the results, which were carried out by Ward, are presented in an appendix to [11]. It is shown that there exists a certain discrepancy between theory and experiment, although the main form of the instability, according to Ward's assertion, agrees with theoretical predictions.

As possible causes of this discrepancy, Ward considered the following effects: the effect of the force of gravity on the liquid filling, causing the axis of rotation to be different from the axis of the cylinder; the effect of the nonlinear terms and the effect of the forces of friction on the gimbal suspension. By experiment it is shown that the effects are so insignificant that they cannot be the cause of this discrepancy. The possibility of the free surface differing sharply from a cylindrical shape is not brought out in this paper [11].

Here we study the characteristic oscillations of the free surface of a liquid in the cylindrical cavity of a gyroscope and their influence on the stability of the whole system.

Let us consider the motion of a weighty symmetrical gyroscope (a top with a fixed point of support) when the centre of mass of the gyroscope lies on the axis of rotation of the ellipsoid of inertia, constructed for the point of support. Let the cavity be a cylinder, the axis of which coincides with the axis of symmetry of the gyroscope (top). In order to study the stability of rotational motion of the partially filled gyroscope we divide the problem into two parts: (a) the motion of the liquid in the cavity, when the gyroscope is rotating about a fixed axis of symmetry; (b) the influence of the liquid on the stability of the gyroscope.

To solve the first part of the problem let us consider two cases: (1) when the velocity of intrinsic rotation of the gyroscope ω is such that $a^2\omega^2 \gg gc$, where a is the radius of the cavity and $2c$ is its height; (2) when ω is small and $a^2\omega^2 \ll gc$.

Let us introduce a moving system of coordinates $Oxyz$: the z-axis is chosen vertically upwards whilst the axes of x and y are in a horizontal plane rotating about z with velocity ω and form a right-handed set with z. Let us denote by $\{u^*, v^*, w^*\}$ the velocities of the fluid particle (x, y, z) along these axes at the instant t. In the case of large velocities of rotation the oscillations arise from the action of centrifugal forces, in the case of small ω from the action of the force of gravity.

1. Waves on the surface of a slowly rotating liquid in a cylindrical vessel. Let a cylindrical vessel with a flat bottom contain liquid and rotate together with the liquid about the vertical axis of symmetry with angular velocity ω. For relative equilibrium under the action only of the force of gravity, the free surface is a paraboloid of revolution. Let us assume that the inclination of the surface is small, i.e. $a\omega^2 \ll g$, where a is the radius of the vessel. The liquid will rotate together with the vessel as a rigid body after a lapse of time of order a^2/ν, where ν is the kinematic coefficient of viscosity of the liquid. Let us assume that in the relative motion the velocities are small, as a result of which we can discard from the equations of motion those terms which are of second order in the relative velocities and also in the force of viscosity (friction).

The equations of motion in the moving coordinates are

$$\frac{\partial u^*}{\partial t} - 2\omega v^* = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \omega^2 x + F_x \tag{1.1}$$

$$\frac{\partial v^*}{\partial t} + 2\omega u^* = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \omega^2 y + F_y, \qquad \frac{\partial w^*}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F$$

Here g is the acceleration due to gravity, $F(F_x, F_y, F_z)$ is the vector of the external body force, p is the pressure of the liquid at the point (x, y, z).

We note that in the eighth chapter of [13] some results are given relating to the oscillations of a horizontal layer of liquid of constant thickness and of variable thickness, having a shape defined by $h = h_0(1 - r^2/a^2)$; the cases studied are when $\omega = 0$ and when $\omega \neq 0$ but $a\omega^2 \ll g$. Let us study the small oscillations performed by a liquid with a free surface under the action of the force of gravity.

Let the xy-plane coincide with the plane of the base. Then in the unperturbed state the thickness of the layer of liquid is a function of $(x^2 + y^2)$, i.e.

$$h = \frac{1}{2}\frac{\omega^2}{g}(x^2 + y^2) + h_0 \tag{1.2}$$

From the total volume of liquid and ω we can easily find h_0. It will be assumed that sufficient liquid is present so that h does not equal zero anywhere under the conditions of rotation ($h_0 \neq 0$), i.e. the paraboloid of revolution does not cut the xy-plane.

Let us denote by $\zeta^*(x, y)$ the elevation of the free surface above the unperturbed level at the point (x, y). We obtain the equation of continuity, calculating the flux of liquid into an elementary prism with the base $\delta x\delta y$; neglecting second-order terms, we obtain [13]

$$\frac{\partial \zeta^*}{\partial t} = -\frac{\partial (u^* h)}{\partial x} - \frac{\partial (v^* h)}{\partial y} \tag{1.3}$$

Assuming that the vertical acceleration of the liquid is small in comparison with g, the pressure of the liquid at the point (x, y, z) is determined by the equation

$$p = p_0 + g\rho(h + \zeta^* - z) \tag{1.4}$$

Here $h + \zeta^*$ is the ordinate of the perturbed surface. It is clear that when $\zeta^* = 0$ in the unperturbed state all the equations of motion are satisfied with $u^* = v^* = w^* = 0$ and $p = p_0 + \rho g(h - z)$.

Let us consider the characteristic oscillations of the layer of liquid under the action of the force of gravity, i.e. in the equations of motion (1.1) let us set $F = 0$. The functions u^*, v^*, w^* and ζ^* are sought in the form

$$u^* = u(x, y, z)\,e^{i\sigma t}, \ \ldots, \ \zeta^* = \zeta(x, y)\,e^{i\sigma t}$$

It is convenient to employ cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The equations of horizontal motion (making use of the relation (1.4)) have the form [13]

$$i\sigma u - 2\omega v = -g\frac{\partial \zeta}{\partial r}, \qquad i\sigma v + 2\omega u = -g\frac{\partial \zeta}{r\partial \theta} \tag{1.5}$$

The equation of continuity in cylindrical coordinates becomes

$$i\sigma\zeta = -\frac{\partial (rhu)}{r\partial r} - \frac{\partial (hv)}{r\partial \theta} \tag{1.6}$$

which by virtue of (1.2) reduces to the form

$$i\sigma\zeta = -h\left\{\frac{\partial (ru)}{r\partial r} + \frac{\partial v}{r\partial \theta}\right\} - u\frac{\partial h}{\partial r} \tag{1.7}$$

From Equations (1.5) we find u and v

$$u = \frac{g}{\sigma^2 - 4\omega^2}\left(i\sigma \frac{\partial\zeta}{\partial r} + 2\omega \frac{1}{r}\frac{\partial\zeta}{\partial\theta}\right), \quad v = \frac{g}{\sigma^2 - 4\omega^2}\left(i\sigma \frac{\partial\zeta}{r\partial\theta} - 2\omega \frac{\partial\zeta}{\partial r}\right) \tag{1.8}$$

and, substituting in (1.7) to find the function ζ, we obtain

$$(\sigma^2 - 4\omega^2)\,\zeta = -hg\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\zeta - \omega^2 r\left(\frac{\partial}{\partial r} - \frac{2\omega i}{r\sigma}\frac{\partial}{\partial\theta}\right)\zeta \tag{1.9}$$

The function $\zeta(r, \theta)$, periodic in θ, may be expanded in a Fourier series in $e^{is\theta}$ with undetermined coefficients depending on r

$$\zeta(r, \theta) = \sum_{-\infty}^{\infty} \zeta_s(r)\, e^{is\theta} \tag{1.10}$$

Substituting (1.10) in (1.9) for each of the functions $\zeta_s(r)$, we obtain

$$\left(1 + \frac{r^2}{\lambda^2}\right)\left(\frac{d^2\zeta_s}{dr^2} + \frac{1}{r}\frac{d\zeta_s}{dr} - \frac{s^2}{r^2}\zeta_s\right) + \frac{2}{\lambda^2}\left(r\frac{d\zeta_s}{dr} + \frac{2\omega s}{\sigma}\zeta_s\right) + \frac{\sigma^2 - 4\omega^2}{gh_0}\zeta_s = 0$$

$$\left(\lambda^2 = \frac{2gh_0}{\omega^2}\right) \qquad (s = 0, \pm 1, \pm 2 \ldots) \tag{1.11}$$

Equation (1.11) can be put in the form

$$\frac{d^2\zeta_s}{dr^2} + p(r)\frac{d\zeta_s}{dr} + q(r)\,\zeta_s = 0 \tag{1.12}$$

Here

$$p(r) = \frac{1}{r} + \frac{2r}{\lambda^2}\left(1 + \frac{r^2}{\lambda^2}\right)^{-1}, \qquad q(r) = \left(\frac{4\omega s}{\lambda^2\sigma} + \frac{\sigma^2 - 4\omega^2}{gh_0}\right)\left(1 + \frac{r^2}{\lambda^2}\right)^{-1} - \frac{s^2}{r^2}$$

The point $r = 0$ is a pole of $p(r)$ and $q(r)$, but $rp(r)$ and $r^2q(r)$ are analytic in the neighborhood of $r = 0$; consequently, the point $r = 0$ is a regular point for the differential equation (1.12). Let us seek the solution of Equation (1.12) in the form of a power series, multiplied by r^a, i.e.

$$\zeta_s = \left(\frac{r}{\lambda}\right)^a \left[1 + \sum_{n=1}^{\infty} a_n \left(\frac{r}{\lambda}\right)^n\right] \tag{1.13}$$

where a_1 and a_2 are constant coefficients. If we define the dimensionless quantity $r/\lambda = \eta$, then Equation (1.12) is

$$\eta^2 \frac{d^2\zeta_s}{d\eta^2} + \eta\left(1 + \frac{2\eta^2}{1 + \eta^2}\right)\frac{d\zeta_s}{d\eta} + \left[-s^2 + \frac{\eta^2}{1 + \eta^2}\left(\frac{4\omega s}{\sigma} + \frac{\sigma^2 - 4\omega^2}{gh_0}\lambda^2\right)\right]\zeta_s = 0 \tag{1.14}$$

Let us expand the coefficients of ζ_s' and ζ_s in series in the interval

$0 \leqslant \eta < 1$

$$1 + \frac{2\eta^2}{1+\eta^2} = 1 + 2\eta^2 - 2\eta^4 + 2\eta^6 - \ldots$$

$$-s^2 + \frac{\eta^2}{1+\eta^2}\delta = -s^2 + \delta(\eta^2 - \eta^4 + \eta^6 - \ldots) \qquad (1.15)$$

$$\left(\delta = \frac{4\omega s}{\sigma} - \frac{2(\sigma^2 - 4\omega^2)}{\omega^2}\right)$$

The dimensionless quantity η will vary in the interval $0 \leqslant \eta < 1$ if $a < \lambda$ or, which is the same thing, if $a^2\omega^2 < 2gh_0$.

Substituting $\zeta_s = \eta^\alpha(1 + a_1\eta + a_2\eta^2 + \ldots)$ in the differential equation (1.14) (assuming that differentiation and multiplication of the series are permissible) and bearing in mind the expansions (1.15), we obtain a power series. Equating to zero the coefficients of successive powers of η, we obtain the following system of equations:

$$F(\alpha) \equiv \alpha^2 - s^2 = 0 \qquad (1.16)$$

$$a_1 = 0, \qquad a_2\{(\alpha+2)^2 - s^2\} + 2\alpha + \delta = 0, \qquad a_3\{(\alpha+3)^2 - s^2\} = 0, \ldots$$

$$a_n\{(\alpha+n)^2 - s^2\} + \sum_{m=1}^{n-1} a_{n-m}\{(\alpha+n-m)p_m + q_m\} + \alpha p_n + q_n = 0, \ldots \qquad (1.17)$$

Here

$$p_0 = 1, \quad p_1 = 0, \quad p_2 = 2, \quad p_{2k+1} = 0, \quad p_{2k+2} = (-1)^{k+1}2$$

$$q_0 = -s^2, \quad q_{2k+1} = 0, \quad q_{2k} = (-1)^{k+1}\delta \qquad (1.18)$$

For each value of α from (1.16) all the coefficients a_1, a_2, ... are determined in succession. We find that the coefficients a_n for odd powers of η vanish, i.e. $a_{2k+1} = 0$. We notice that $\alpha_1 - \alpha_2 = 2s$ is a whole number, and accordingly we choose only the first exponent $\alpha_1 = s$.

The second linearly independent solution of the differential equation contains $\ln \eta$, and accordingly it will not be considered.

Substituting α in the system (1.17), we find

$$a_2 = -\frac{2s+\delta}{2(2s+2)}, \qquad a_4 = \frac{(2s+\delta)(6s+\delta+8)}{2\cdot 4(2s+2)(2s+4)} \qquad (1.19)$$

Then a_6 is expressed in terms of a_2 and a_4, and so on. From (1.17) and (1.18) we can obtain the relation between successive coefficients:

$$4(k+1)(s+k+1)a_{2k+2} + [(s+2k)2 + \delta + 4k(s+k)]a_{2k} = 0 \quad (1.20)$$

Accordingly, we obtain a recurrence formula for the determination of the coefficients of each power of η.

Let us prove the convergence of the expansion (1.13) for the actual values of $a_k (k = 1, 2, 3, \ldots)$.

Remembering that the coefficients of odd powers of η are zero, and making use of the relation (1.20), we obtain

$$\lim_{k\to\infty} \frac{|a_{2k+2}|}{|a_{2k}|} = \lim_{k\to\infty} \frac{|4k(s+k)+\delta+2(s+2k)|}{4(k+1)(s+k+1)} = 1$$

The series (1.13) therefore converges if $|\eta| < 1$. Accordingly, in the interval $|\eta| < 1$ it is permissible to differentiate and multiply the series (1.13) and (1.15). The solution so constructed in the form of a power series, namely

$$\zeta_s = \left(\frac{r}{\lambda}\right)^s \left[1 + \sum_{k=1}^{\infty} a_{2k} \left(\frac{r}{\lambda}\right)^{2k}\right] \tag{1.21}$$

converges for all values of $r (0 \leqslant r \leqslant a)$. In (1.21) the coefficients a_{2k} depend upon s and δ. The solutions which are found must satisfy the boundary condition

$$u = 0 \text{ when } r = a \tag{1.22}$$

i.e. on the lateral surface of the cylinder the radial component of velocity of the liquid must vanish. Bearing in mind (1.8), (1.10) and (1.21), we find that the condition (1.22) reduces to the equation

$$f\left(\frac{\sigma}{\omega}\right) \equiv \frac{a}{\lambda} \zeta_s'\left(\frac{a}{\lambda}\right) + \frac{2\omega s}{\sigma} \zeta_s\left(\frac{a}{\lambda}\right) = 0 \tag{1.23}$$

Hence the frequency of oscillation of the free surface is also determined.

Numerical example. To find the root of Equation (1.23) in the particular case when $a/\lambda = 1/2$, a table of values of $f(\sigma/\omega)$ in the range $-10 \leqslant \sigma/\omega \leqslant 10$ was constructed.

The values of the function f were obtained at 160 points of the range $(-10, 10)$. The distance between two successive points was equal to 1/8. The following results were obtained:

when $s = 0$ the function $f(\sigma/\omega)$ does not change sign in the specified interval, i.e. there is no root;

when $s = 1$ the function has roots in the intervals

$$\left(-\tfrac{3}{8}, -\tfrac{2}{8}\right), \quad \left(\tfrac{1}{8}, \tfrac{2}{8}\right), \quad \left(5\tfrac{8}{7}, 6\right)$$

when $s = 2$ the function has roots in the intervals

$$\left(-7\tfrac{5}{8}, \ -7\tfrac{4}{8}\right), \quad \left(-\tfrac{4}{8}, \ -\tfrac{3}{8}\right), \quad \left(\tfrac{2}{8}, \tfrac{3}{8}\right), \quad \left(\tfrac{4}{8}, \tfrac{5}{8}\right), \quad \left(2\tfrac{2}{8}, \ 2\tfrac{1}{4}\right)$$

2. Oscillations of the free surface of the liquid with large angular velocities of rotation of the container. 1. Let us consider oscillations of the free surface of the liquid in a cylindrical vessel which is rotating rapidly about a vertical axis.

In the unperturbed state we shall assume rotation about the vertical axis of symmetry of the whole system as a rigid body, i.e.

$$u_r = 0, \qquad u_\theta = r\omega, \qquad u_z = 0$$
$$p = \tfrac{1}{2}\rho\omega^2 (r^2 - b^2) + g\rho (z - c) + p_0 \tag{2.1}$$

Here $\{u_r, u_\theta, u_z\}$ is the velocity of a fluid particle, p is the pressure, whilst p_0 is the pressure of the overlying air, b is the radius of the free surface. In the perturbed state the radius of the free surface will be denoted by $b + \zeta^*(\theta, z)$. Let us assume that during the perturbed motion the velocity of the relative motion does not influence the pressure p; let us determine p from the static conditions, taking account of the change of the free surface:

$$p = p_0 + \tfrac{1}{2}\rho\omega^2 (r^2 - b^2) - \rho\omega^2 b\zeta^* (\theta, z) + \rho g (z - c) \tag{2.2}$$

If we write down the equation of relative motion in cylindrical coordinates and substitute the value of p from (2.2), we obtain

$$\frac{\partial u^*}{\partial t} - 2\omega v^* = F_r, \quad \frac{\partial v^*}{\partial t} + 2\omega u^* = \omega^2 \frac{\partial \zeta^*}{\partial \theta} + F_\theta, \quad \frac{\partial w^*}{\partial t} = \omega^2 b \frac{\partial \zeta^*}{\partial z} + F_z \tag{2.3}$$

Here (u^*, v^*, w^*) are the components of velocity in cylindrical coordinates.

We obtain the equation of continuity by calculating the flux of fluid into an elemental pyramidal volume formed by the coordinate planes z, $z + \delta z$, θ, $\theta + \delta\theta$, the surface of the cavity and the free surface. Neglecting terms of the second order, we obtain

$$\frac{\partial}{\partial \theta}(hv^* \delta z)\, \delta\theta + \frac{\partial}{\partial z}(r_0 h w^* \delta\theta)\, \delta z = \frac{\partial}{\partial t}\left[(b + \zeta^*)\, b\delta\theta\, \delta z\right] \tag{2.4}$$

Hence

$$b\frac{\partial \zeta^*}{\partial t} = \frac{\partial}{\partial z}(r_0 h w^*) + \frac{\partial}{\partial \theta}(hv^*) \tag{2.5}$$

Here r_0 is the mean radius of the horizontal section, assuming the cavity to have the form of a body of revolution; h is the thickness of the layer of liquid. The values of r_0 and h are taken from the unperturbed state.

The functions u^*, v^*, w^* and ζ^* will be sought in the form

$$u^* = ue^{i\sigma t},\ldots \qquad \zeta^* = \zeta(\theta, z)e^{i\sigma t} \tag{2.6}$$

Substituting (2.6) in the equation of motion (2.3) and neglecting the forces F_r, F_θ, F_z, we obtain

$$i\sigma u - 2\omega v = 0, \qquad i\sigma v + 2\omega u = \omega^2 \frac{\partial \zeta}{\partial \theta}, \qquad i\sigma\omega = \omega^2 b \frac{\partial \zeta}{\partial z} \tag{2.7}$$

Hence

$$v = \frac{i\sigma\omega^2}{4\omega^2 - \sigma^2}\frac{\partial \zeta}{\partial \theta}, \qquad w = \frac{\omega^2 b}{i\sigma}\frac{\partial \zeta}{\partial z} \tag{2.8}$$

In the general case, when the cavity is a body of revolution, the axis of symmetry of which coincides with the axis of rotation of the container, the thickness of the layer of liquid in the unperturbed state depends only on the coordinate z. The equation determining the form of the free surface in the perturbed state can be obtained by substituting (2.8) in (2.5) and remembering that $h = h(z)$ and $r_0 = r_0(z)$. It will have the form

$$h(z)\, r_0(z)\frac{\partial^2 \zeta}{\partial z^2} + \frac{d}{dz}[(h(z)\, r_0(z)]\frac{\partial \zeta}{\partial z} + \frac{h(z)}{b}\frac{\sigma^2}{\sigma^2 - 4\omega^2}\frac{\partial^2 \zeta}{\partial \theta^2} + \frac{\sigma^2}{\omega^2}\zeta = 0$$

Let us consider the case of a cylindrical cavity with radius a and height $2c$. In this case h and r_0 do not depend on z, and the equation describing the variation of the free surface of the liquid is

$$\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma^2}{br_0(\sigma^2 - 4\omega^2)}\frac{\partial^2 \zeta}{\partial \theta^2} + \frac{\sigma^2}{hr_0\omega^2}\zeta = 0 \tag{2.9}$$

The function ζ must satisfy the following boundary condition:

$$\partial\zeta/\partial z = 0 \text{ when } z = \pm c \tag{2.10}$$

i.e. the w-component of velocity of the liquid particles at the ends of the cylinder must vanish. In order to show the dependence of ζ on θ let us expand ζ as a Fourier series in cosines and sines of the azimuth θ, or, which is the same thing, in $e^{is\theta}$, where $s = 0, \pm 1, \pm 2, \ldots$. We obtain a series, the terms of which have the form $f_s(z)e^{is\theta}$.

Substituting the expansion of ζ in Equation (2.9) and equating to zero the coefficient of each power $e^{is\theta}$, we obtain

$$\frac{d^2f_s}{dz^2}+\frac{1}{hr_0}\left(\frac{\sigma^2}{\omega^2}-\frac{h}{b}\frac{\sigma^2s^2}{\sigma^2-4\omega^2}\right)f_s=0 \tag{2.11}$$

Let us write the general solution of Equation (2.11) in the form

$$f_s(z)=A_s\cos(\mu z+\varepsilon),\qquad \mu^2=\frac{1}{r_0h}\left(\frac{\sigma^2}{\omega^2}-\frac{h}{b}\frac{\sigma^2s^2}{\sigma^2-4\omega^2}\right) \tag{2.12}$$

Here A_s is an arbitrary constant.

Possible values of σ are determined from the condition (2.10), i.e.

$$\sin(\pm\mu c+\varepsilon)=0 \tag{2.13}$$

Hence, to determine the frequency of oscillation of the free surface we obtain the formula

$$\frac{1}{r_0h}\left(\frac{\sigma^2}{\omega_2}-\frac{h}{b}\frac{\sigma^2s^2}{\sigma^2-4\omega^2}\right)=\frac{k^2\pi^2}{4c^2} \tag{2.14}$$

Let us introduce the following notation:

$$\frac{a^2}{c^2}=\varkappa,\qquad 1-\frac{b^2}{a^2}=\eta,\qquad \left(\frac{\sigma}{\omega}\right)^2=\lambda \tag{2.15}$$

Then Equation (2.14) takes the form

$$\varphi(\lambda)\equiv\lambda^2-\lambda\left[4+s^2\left(\frac{1}{\sqrt{1-\eta}}-1\right)+\frac{k^2\pi^2}{8}\varkappa\eta\right]+\frac{k^2\pi^2}{2}\eta\varkappa=0 \tag{2.16}$$

In Equation (2.16) the free term is positive, whilst at the minimum point

$$\lambda_1=\frac{1}{2}\left[4+\left(\frac{1}{\sqrt{1-\eta}}-1\right)s^2+\frac{k^2\pi^2}{8}\varkappa\eta\right] \tag{2.17}$$

the value of the function $\phi(\lambda_1)$ is less than or equal to zero, i.e.

$$\varphi(\lambda_1)=-\frac{1}{4}\left[4+\left(\frac{1}{\sqrt{1-\eta}}-1\right)s^2+\frac{k^2\pi^2}{8}\varkappa\eta\right]^2+\frac{k^2\pi^2}{2}\varkappa\eta\leqslant$$
$$\leqslant-\left(2-\frac{1}{16}k^2\pi^2\varkappa\eta\right)^2\leqslant 0$$

Hence it follows that all the roots of Equation (2.16) are real and positive for any k, s, η and κ. Consequently, all four roots of Equation (2.14) are real.

In the case $s = 0$ the free surface has the form of a body of revolution, and for σ we obtain

$$\sigma=\pm\omega\frac{k\pi}{2}\sqrt{\frac{\varkappa\eta}{2}}$$

When $k = 0$ the free surface does not change in the direction of the axis of the cavity, and for σ we obtain

$$\sigma = \pm \omega \{4 + [(1-\eta)^{-\frac{1}{2}} - 1]s^2\}^{\frac{1}{2}}$$

In the general case σ is determined from the formula

$$\sigma = \pm \omega \left\{\frac{A+B}{2} \pm \left[\left(\frac{A+B}{2}\right)^2 - 4B\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\left(A = 4 + [(1-\eta)^{-\frac{1}{2}} - 1]\, s^2,\ B = \frac{k^2\pi^2}{8}\eta\varkappa\right) \tag{2.18}$$

In this way, the frequencies of characteristic oscillations of the liquid in a cylindrical cavity are determined as a function of its fullness η.

For characteristic oscillations of the free surface of a liquid we have

$$\zeta^* (\theta, z, t) = A_s \cos (\mu z + \varepsilon) \exp [i (s\theta + \sigma t)] \tag{2.19}$$

For each fixed value of t the equation of the free surface $r = b + \zeta^* (\theta, z, t)$ takes the form of a cylinder with a skewed axis, except in the case when $s = 0$ or $k = 0$. In [14] Malashenko describes experimental investigations relating to bodies of revolution. In particular, he photographed a rotating transparent body containing liquid in a cavity, i.e. he captured an instantaneous picture of the motion of the free surface of the liquid.

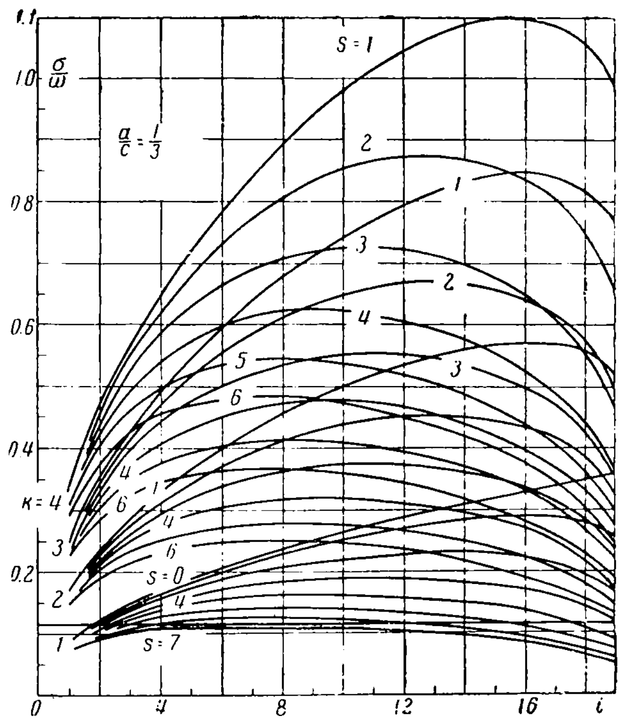

Fig. 1.

The form of the free surface for a fixed value of t, determined according to Formula (2.19), is verified by the picture of the free surface in the model with the transparent cylindrical cavity introduced in Malashenko's paper.

2. By way of illustration, we shall present the results of a computation (on the electronic computing machine "Erivan") of the roots of Equation (2.14) for three cylindrical cavities defined by the parameters $a/c = 1/3,\ 2/3,\ 1$. In each case the roots were calculated for values of $k = 0,\ 1,\ 2,\ 3,\ 4$; $s = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$, whilst η varies with increment $\Delta\eta = 0.05$, i.e. $\eta = 0.05\ i$, where i takes integral values from 1 to 19.

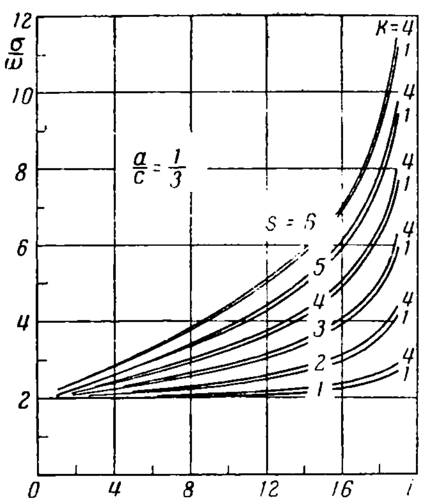

Fig. 2..

From the computed values we constructed graphs showing the dependence of the absolute values of the frequency of oscillation σ on the filling coefficient η, for various values of k and s.

In Figs. 1 and 2 we show this dependence for both values of σ from (2.20) in the case $a/c = 1/3$. Figure 1 refers to the smaller values (the minus sign in front of the square bracket), whilst Fig. 2 refers to the larger (plus sign). The same dependence for the cases $a/c = 2/3$ and $a/c = 1$ are displayed in Figs. 3, 4 and 5, 6, respectively.

3. The influence of the liquid oscillations on the stability of the gyroscope. In the theory of the gyroscope [15] it is well known that the most general form of motion of a weighty symmetric gyroscope, possessing great intrinsic angular momentum, is a pseudo-regular precession about a vertical line passing through the point of support. For a fast gyroscope with pseudo-regular precession the velocity of nutation (the velocity with which the axis of the top rotates about the axis of angular momentum) does not depend upon the overturning moment and is directly proportional to the angular momentum. For a rapid intrinsic rotation the following relation is approximately true: $\omega_n = \omega C/A$, where ω is the intrinsic angular velocity, ω_n is the angular velocity of nutation, A and C are respectively the equatorial and axial moments of inertia of the gyroscope. If the gyroscope is elongated the nutation is slower, whilst if it is flattened it is faster than the intrinsic rotation of the gyroscope, and moreover the direction of rotation coincides with the direction of the intrinsic rotation. From the relation $\omega_n = \omega A/C$ it follows that the ratio of the velocity of nutation to the velocity of the intrinsic rotation remains constant for each gyroscope. A gyroscope with a cylindrical cavity partially filled with liquid becomes unstable for certain filling ratios and remains stable for other filling ratios [11, 15]. Let us study the dependence of

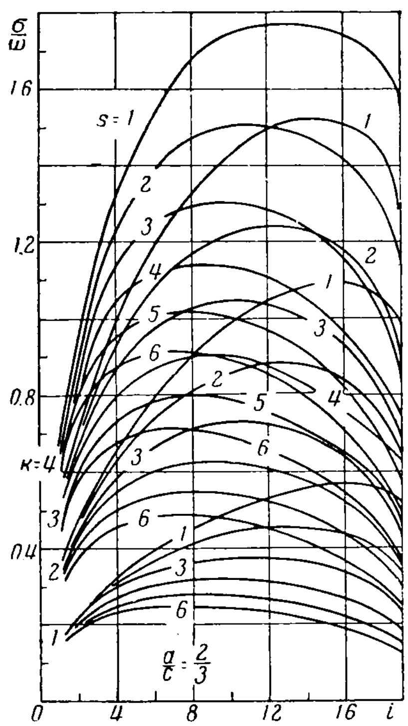

Fig. 3.

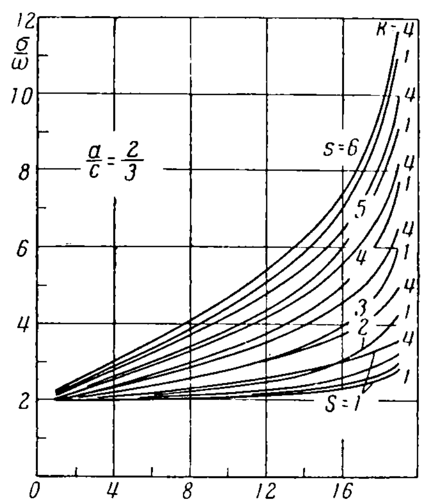

Fig. 4.

the stability of the gyroscope on the amount of liquid filling.

For the undisturbed state of the system – rigid container and liquid – we assume steady rotation about a vertical axis as a rigid body. For small relative motions of the liquid it increases the overturning moment but does not influence the frequency of nutation of the gyroscope (the wobbling of the axis of the body), since the frequency of nutation does not depend on the overturning moment. Consequently, if the liquid in the cavity is not strongly perturbed, the ratio ω_n/ω remains constant. Assuming that the liquid is acted on only by the nutational oscillations of the system, the appearance of instability can be explained in the following way. If for a given filling ratio η the frequency of characteristic oscillations of the free surface of the liquid coincides with the frequency of nutation, then there occurs a strong perturbation of the free surface, which assumes the shape of a cylinder with a skewed axis, i.e. on the free surface of the liquid there appears an asymmetric wave motion, after which the stable state of the gyroscope is upset.

In order to verify the stability of the gyroscope with a given liquid filling ratio, let us compare the values of σ/ω, calculated according to Formula (2.14), with the ratio ω_n/ω. If at least one value of σ/ω coincides with $\omega_n/\omega = C/A$ or becomes sufficiently close to this number, then for such a filling ratio the gyroscope loses stability.

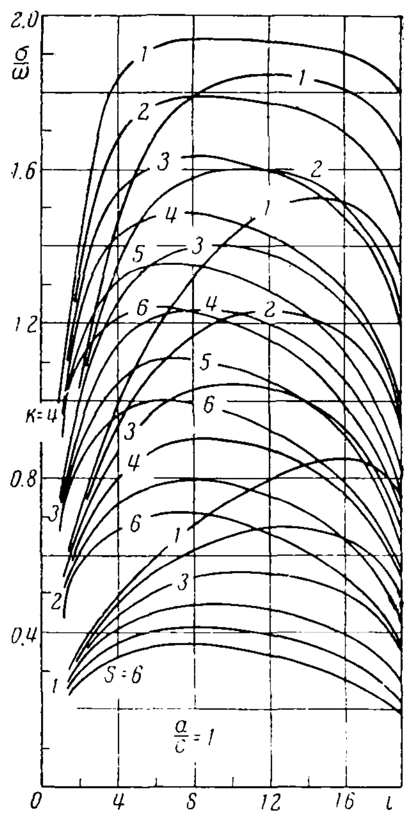

Fig. 5.

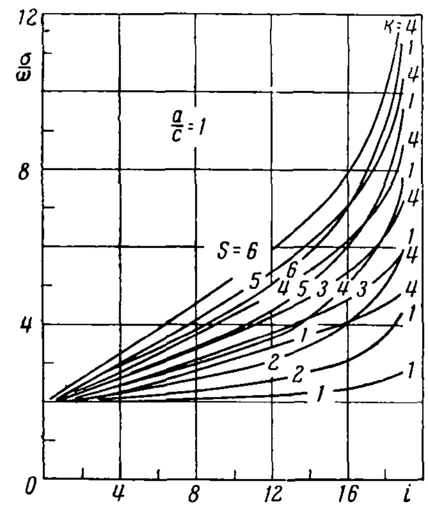

Fig. 6.

Such an approach is vindicated by the experiment carried out by Ward [11]. In these experiments the dimensions of the cavity (in inches) were $2a = 1\ 1/8$ and $2c = 3\ 3/8$, i.e. $a/c = 1/3$, the ratio of the nutational frequency of the gyroscope to its rotational frequency $\omega_n/\omega = A/C = 0.112$, whilst the velocity of rotation $\omega = 6000$ revolutions per minute. The gyroscope was found to be unstable for filling ratios between 0.63 and 0.70.

As is clear from Fig. 1, the graph of the function $\sigma(\eta)/\omega$ with $s = 6$ and $k = 1$ in the interval $0.57 < \eta < 0.68$ and in the vicinity of $\eta = 0.20$ just about coincides with the line $\sigma/\omega = 0.112$ (the bold horizontal line on Fig. 1). When $k = 1$ this line also cuts the curves $s = 1, 2, 3, 4, 5$, but for $\eta < 0.15$. The filling coefficient $\eta = 0.15$ in the experiments [11] corresponds to 6 gm of liquid, whilst the rigid part (the container) of the gyroscope had a greater mass; consequently, when $\eta < 0.15$ the liquid could not influence the stability of the container. Again at two points, namely at $\eta \approx 0.80$ the curve $s = 5$ when $k = 1$, and at $\eta \approx 0.89$ the curve $s = 4$ when $k = 1$ intersect the specified straight line, but at an appreciable angle. In order that instability of the gyroscope be observed, the filling coefficient must coincide exactly with the value $\eta \approx 0.80$ or $\eta \approx 0.89$, since even for fillings very close to these values the corresponding values of σ/ω differ considerably from 0.112.

In the cases $a/c = 2/3$ and $a/c = 1$ (Figs. 3 and 5) the curves of frequency do not intersect the straight line $\sigma/\omega = 0.112$, and the values of σ/ω increase with increasing a/c.

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